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Effects of particle shape on electromagnetic torques: A comparison of the effective-dipole-moment method with the Maxwell-stress-tensor method

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Two methods for calculating the torque exerted by a rotating field upon a particle have emerged: (a) The effective-dipole-moment method and (b) the Maxwell-stress-tensor method. It has previously been assumed that in nonlossy systems, both methods will yield identical results. In this paper we show that significant differences appear depending upon the particle shape.

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I. INTRODUCTION

The development of physical methods that are capable of probing the properties of intact biological cells is an active topic of current experimental and theoretical interest. Among the rich variety of methods currently being developed are two well-known ones: the dielectrophoretic (DEP) levitation method and the method of electrorotation. Both these methods afford, in a complementary manner, information regarding the dielectric properties of the cell. The theoretical analysis underlying this type of experimentation has been discussed in considerable detail, and an extensive list of background references has been provided in a recent paper by Wang, Pethig, and Jones [1]. These authors show that the two experimental methods stated above are, respectively, sources of information for the real and imaginary parts of the well-known Claussius-Mossotti factor. Descriptions of some of the advances in the experimental technology can be found in several recent papers, for example, a combination of both dielectrophoresis and electrorotation have been used in a recent experimental study carried out by Fuhr *et al.* [2]. In this paper we focus our attention on one of the aspects within the theory of electrorotation of nonspherical cells (in particular those of ellipsoidal shape) which appears to have been neglected in all discussions of the subject. It is well known that electrorotation is the rotational motion of the cell resulting from the torque that is exerted upon it by the field. As a consequence of this the theoretical study of the origin of the torque has been of some interest.

An examination of the literature in this field shows that two methods of calculating the torque have evolved.

(1) *The effective-dipole-moment method* in which the effective dipole moment induced on the cell by the external field is first computed and then the torque exerted on this (the dipole) by the field is calculated by using standard formulas of classical mechanics. This approach is a very elegant one and has been developed into a successful tool largely due to the recent efforts of Miller and Jones [3].

(2) *The Maxwell-stress-tensor method* in which the force per unit area exerted on the surface of the cell by the electromagnetic field is calculated by using the Maxwell stress tensor and the cross product of this surface force density with the radial vector integrated over the entire cellular surface yields the torque. The details of the derivations of the general equations for this method can be found in any standard textbook on electromagnetism, for example, in the well-known book by Landau and Lifshitz [4].

Of the two methods listed above the second one is quite cumbersome since it involves the computation of surface integrals that may be quite difficult to obtain for certain geometrical shapes. As a consequence of this much of the work carried out in the field has employed the first method of calculation. Recently, however, Sauer [5,6] has raised some serious doubts about the validity of this approach for lossy systems (The meaning of this term is often confused. As far as this paper is concerned, by a "lossy" system we mean a system in which the dipoles, which are either permanently present or have been induced by the external field, are unable to keep up with the oscillations of the external ac field and thus cause a dissipation of energy to occur.) Sauer has insisted that the second exact approach must be used in such cases. In Ref. [5] the main burden of the paper has been concerned with the calculation of the force exerted on the center of mass of a particle using the stress-tensor approach and an attempt has been made to show the differences of this method from the more conventional methods of computing the ponderomotive force, for example, that due to Sher [7]. Unfortunately a similar comparison of the torque is not easy to make since the formalism tends to get fairly cumbersome in the Maxwell-stress-tensor approach. It has been tacitly assumed that the two methods will lead to identical results and that differences will only be observed when dealing with "lossy" systems. In this paper we show that even if a dc rotating field is used with a nonlossy ellipsoidal particle the two methods are not identical and the results are strongly dependent upon the particle geometry. Only in the case of spherical particles do the results from the two approaches become identical.

II. CALCULATIONS

Since most cells that are encountered in nature are, in general, nonspherical in shape we will select, for the purposes of this paper, an ellipsoidal cell with a homogeneous internal structure. The three axes will be taken to be $a > b > c$ with the largest axis a oriented along the x axis of a Cartesian frame and a frequency-independent electrical external field rotating in the yz plane as shown in Fig. 1. The cell will be assumed to be suspended in a liquid medium with real dielectric constant ϵ_l.

We now consider the torque as calculated by the two methods.

A. The effective-dipole-moment method

According to this method the torque can be found by using a well-known formula of classical mechanics that equates the torque experienced by a dipole in an external field to the cross product of the dipole moment $\mathbf{p}$ (in the present case an induced dipole moment) and the applied field $\mathbf{E}_0$:

$$\mathbf{T}=\mathbf{p}\times\mathbf{E}_0 \ . \tag{1}$$

This equation has been generalized by Miller and Jones [3] for lossy systems by considering the time averages of the corresponding expression that is obtained when all the quantities are replaced by their complex counterparts. For the purposes of this paper we will consider real fields only. In a well-known textbook on electromagnetism Stratton [8] has derived an expression for the potential that would be observed outside an ellipsoidal particle in which a dipole has been induced by the application of an external field. With the geometrical arrangement shown in Fig. 1 this potential expression may be utilized to extract the form of the induced dipole moment that is the cause of the potential. Written in terms of its components this induced dipole moment is given by

$$p_a=\frac{4\pi abc}{3}\left[\frac{\epsilon_s-\epsilon_l}{1+\frac{abc}{2}\left[\frac{\epsilon_s}{\epsilon_l}-1\right]A_a}\right]E_{0a} \ , \quad a=x,\ y,\ \text{or}\ z \ . \tag{2}$$

Here, ϵ_s is the dielectric constant of the particle, ϵ_l is the dielectric constant of the surrounding liquid, and A_a is an integral that is defined as follows:

$$A_a=\int_0^{\infty}\frac{ds}{(s+a^2)\sqrt{(s+a^2)(s+b^2)(s+c^2)}} \ , \quad a=a,\ b,\ \text{or}\ c \ . \tag{3}$$

Once these quantities have been defined the torque can be very easily calculated using the standard formula given above.

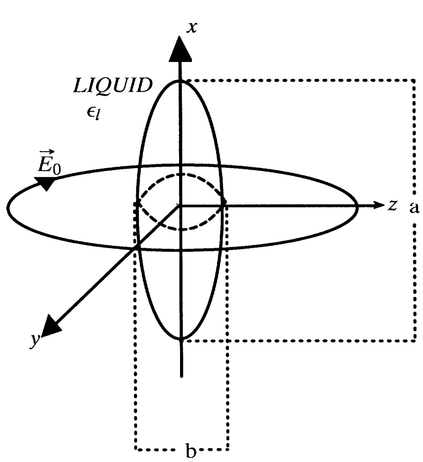

FIG. 1. Diagram displaying the coordinate frame and the rotating field in the yz plane.

B. The Maxwell-stress-tensor method

The basic advantage of this method lies in the fact that it utilizes the Poynting vector which is a quantity that remains invariant in mathematical form whether or not the medium under consideration is lossy. This nature of the Poynting vector ensues from the fact that the tangential component of electrical fields retain their continuity across all surfaces (irrespective of the nature of the materials on either side). A discussion regarding the invariance properties of the Poynting vector may be found in Sec. 80 of Ref. [4].

According to the classical theory of electromagnetism, an electromagnetic field is described by the four classical vectors $\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{H}$. The precise definitions of these vectors may be found in any standard textbook on electromagnetism, for example Ref. [8]. It is also well known that these fields must satisfy equations of motion commonly referred to as Maxwell's equations and, consequently, the equation of motion of any other quantity dependent upon them may be derived by the application of these standard differential equations. As mentioned above, the electromagnetic momentum density or Poynting vector (for which we employ the symbol $\mathbf{p}_e$) is a quantity whose definition remains independent of the nature of the system and in order to compute the surface force density we require its equation of motion.

We begin by writing down the explicit form of $\mathbf{p}_e$ in terms of the four basic vectors of the electromagnetism and then we endeavor to compute the rate of change of $\mathbf{p}_e$ which is equal to the force experienced by the cell

$$\mathbf{p}_e=\left[\frac{1}{c^2}\right](\mathbf{E}\times\mathbf{H}) \ . \tag{4}$$

Here, c is the speed of light. If we differentiate $\mathbf{p}_e$ with respect to the time (t) we will obtain time derivatives of both the electrical and magnetic fields which in turn may be expressed in terms of Maxwell's equations and the final result written in a form of a conservation law. The precise details of this calculation are fairly tedious and are also well understood. They have been discussed in a more general context by deGroot and Mazur [9] and within the confines of the spherical version of the present problem by Sauer [5,6]. In view of this we will content ourselves with the final result:

$$\frac{\partial \mathbf{p}_e}{\partial t}=\nabla\cdot\overleftrightarrow{\sigma} \ . \tag{5}$$

Here, the quantity $\overleftrightarrow{\sigma}$ is called the Maxwell stress tensor and is defined as

$$\overleftrightarrow{\sigma}=\left[\frac{\epsilon_l}{4\pi}\right]\left[\mathbf{EE}-\frac{1}{2}|\mathbf{E}|^2\overleftrightarrow{1}\right] , \tag{6}$$

where $\overleftrightarrow{1}$ is the unit tensor. We now proceed to employ Eq. (6) to calculate the torque density τ. In order to achieve this end we consider the angular momentum density $\mathcal{L}$ given by

$$\mathcal{L}=\mathbf{r}\times\mathbf{p}_e \ , \tag{7}$$

where $\mathbf{r}$ is a vector that is directed from the center of mass of the cell to a point where the angular momentum is required. The torque density is obtained by calculating the time derivative of the angular momentum density vector and utilizing Eqs. (4) and (6). The total torque is then given by the volume integral of the density. The final formula may be expressed in terms of an integral over the cellular surface and is given by

$$\mathbf{T}=\left[\frac{\epsilon_l}{4\pi}\right]\int da_s\left[\mathbf{r}\times\mathbf{E}(\mathbf{E}\cdot\mathbf{n}_s)-\frac{1}{2}|\mathbf{E}|^2(\mathbf{r}\times\mathbf{n}_s)\right] , \tag{8}$$

where da_s is a surface element of the cell and $\mathbf{n}_s$ is a unit normal. In Eq. (8) the field $\mathbf{E}$ is the field that resides *outside the cell.*

From Eq. (8) we see that the first quantity that must be obtained is the field $\mathbf{E}$ outside the cell. In order to achieve this we consider a primary field due to a charge density located upon suitable conductors that function as electrodes and furthermore we assume that the remainder of space, both within and without the cell, is charge free. Thus the potential may be obtained by solving the Laplace's equation with appropriate boundary conditions. This equation being

$$\nabla^2\psi=0 \ , \tag{9}$$

where ψ is the potential function. The solution of Eq. (9) must satisfy the following boundary conditions.

(i) At infinity the field due to this potential ψ must be identical with the field $\mathbf{E}_0$ due to the electrodes *alone.*

(ii) At the surface of discontinuity between the ellipsoidal cell and the surrounding medium the tangential components of the field inside and outside must be continuous:

$$\mathbf{n}_s\times\mathbf{E}=\mathbf{n}_s\times\mathbf{E}^- \ , \tag{10}$$

where $\mathbf{E}^-$ and $\mathbf{E}$ are the electrical fields inside and outside the cell, respectively.

(iii) The normal components of the displacement vectors within and without must display a difference equal to the charge density which is assumed to be zero in the present case:

$$\epsilon_l(\mathbf{n}_s\cdot\mathbf{E})-\epsilon_s(\mathbf{n}_s\cdot\mathbf{E})=0 \ . \tag{11}$$

Since the cell we are considering is of ellipsoidal shape it is convenient to first transform Eq. (8) into ellipsoidal coordinates and then obtain solutions within this frame. Fortunately much of this work has already been done by Hobson [10] and a detailed account thereof may be found in his well-known textbook. We will content ourselves with the final results. Within an ellipsoidal coordinate frame of reference a point (x,y,z) in a Cartesian-coordinate system is expressed in terms of three new variables (ρ,θ, and ϕ). These two sets of coordinates are related to each other as follows:

$$\begin{aligned}&x=\rho\cos\theta \ ,\\&y=(\rho^2-h^2)^{1/2}\sin\theta\cos\phi \ ,\\&z=(\rho^2-k^2)^{1/2}\sin\theta\sin\phi \ ,\\&k^2\le\rho^2<\infty,\quad h^2\le k^2,\quad h>0 \ ,\\&0\le\theta\le\pi,\quad 0\le\phi\le 2\pi \ .\end{aligned} \tag{12}$$

It is easy to see the close similarities with the more familiar polar spherical coordinate system. If, indeed, we set $h=k=0$ in Eq. (12) we immediately recover a sphere with radius ρ. It follows that the parameters h and k are related to the departure of an ellipsoid from a sphere. This feature may be clearly seen, if we consider the following equations to hold with regards to the ellipsoid under consideration:

$$h^2=a^2-b^2,\quad k^2=a^2-c^2 \ . \tag{13}$$

In an ellipsoidal coordinate frame any point in space lies on an ellipsoidal surface. The surface of the cell itself is one such surface characterized by the parameters a,b, and c. This special ellipsoid (the cell surface) will be referred to as the *fundamental ellipsoid* in order to distinguish it from all the other surfaces used for characterizing points either inside or outside the cell.

The Laplace equation may be solved by using the method outlined by Hobson [10] in terms of the so-called ellipsoidal harmonics. These functions play the same role as the spherical harmonics in a polar coordinate frame. By taking the negative gradient of the potential ψ with the gradient operator ∇ expressed in ellipsoidal coordinates the field $\mathbf{E}$ may be obtained and on the surface of the fundamental ellipsoid we have

$$\mathbf{E}=\mathbf{E}_0-\frac{3}{8\pi\epsilon_l}A\cdot\mathbf{p}+\frac{1}{V\epsilon_l}\mathbf{n}_s\mathbf{n}_s\cdot\mathbf{p} \ . \tag{14}$$

Here, V is the volume of the ellipsoid and A is a diagonal 3×3 matrix in which the diagonal elements are the three integrals defined in Eq. (3):

$$A=\begin{bmatrix}A_1&0&0\\0&A_2&0\\0&0&A_3\end{bmatrix} . \tag{15}$$

In the case of a sphere of radius a the three components are equal:

$$A=\frac{2}{3a^3}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} . \tag{16}$$

The vector $\mathbf{p}$ is the effective dipole moment vector defined by Eq. (2).

Two important quantities, $\mathbf{n}_s$ and da_s in Eq. (8), still remain to be defined. Fortunately both these quantities are obtainable by the application of standard methods of differential geometry and a good account is available in Chap. I of Ref. [8] and the results are as follows:

$$\mathbf{n}_s=\frac{a}{h_\rho}[\mathbf{i}(1/a)n_x+\mathbf{j}(1/b)n_y+\mathbf{k}(1/c)n_z] ,$$

$$n_x=\cos\theta,\quad n_y=\sin\theta\cos\phi,\quad n_z=\sin\theta\sin\phi ,$$

$$h_\rho=a\left[\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta\cos^2\phi}{b^2}+\frac{\sin^2\theta\sin^2\phi}{c^2}\right]^{1/2} , \tag{17}$$

$$da_s=bc\,\sin\theta h_\rho d\theta\, d\phi .$$

The quantity h_ρ is often referred to as a scale factor and is related to the metric tensor of differential geometry.

In the next stage of the analysis we substitute Eqs. (14) and (17) into the general torque equation (8) and the result may be expressed in a compact form if the following definitions are employed:

$$\mathbf{x}_1\equiv\mathbf{r}\times\mathbf{E}_0 , \tag{18a}$$

$$x_2\equiv\mathbf{n}_s\cdot\mathbf{E}_0 , \tag{18b}$$

$$\mathbf{x}_3\equiv\mathbf{r}\times\mathbf{n}_s , \tag{18c}$$

$$x_4\equiv\mathbf{n}_s\cdot A\cdot\mathbf{p} , \tag{18d}$$

$$x_5\equiv\mathbf{E}_0\cdot A\cdot\mathbf{p} , \tag{18e}$$

$$x_6\equiv\mathbf{n}_s\cdot\mathbf{p} , \tag{18f}$$

$$x_7\equiv\mathbf{r}\times A\cdot\mathbf{p} , \tag{18g}$$

$$\mathbf{x}_8\equiv \mathbf{A}\cdot\mathbf{p} , \tag{18h}$$

$$\mathbf{r}\equiv\mathbf{i}an_x+\mathbf{j}bn_y+\mathbf{k}cn_z . \tag{19}$$

Here, $\mathbf{r}$ is the radial vector from the center of mass to the surface of the ellipsoid. This vector plays a very crucial role in determining the role played by the actual geometry of the cell as far as the torque is concerned. This fact may be easily recognized if a comparison is made between the vectors $\mathbf{n}_s$, defined in Eq. (17), and $\mathbf{r}$. For cellular shapes where these two vectors are parallel all terms involving their cross products in the set of Eqs. (18) must vanish. This will indeed be the situation in the case of a sphere where the three axes a,b, and c are equal. The torque may now be expressed with Eq. (18) as

$$\mathbf{T}=\frac{\epsilon_l}{4\pi}\int da_s\left[\mathbf{x}_1x_2-\frac{1}{2}E_0^2\mathbf{x}_3-\frac{3}{8\pi\epsilon_l}(\mathbf{x}_1x_4-x_5\mathbf{x}_3+\mathbf{x}_7x_2)+\frac{1}{V\epsilon_l}\mathbf{x}_1x_6+\frac{9}{64\pi^2\epsilon_l^2}\left[\mathbf{x}_7x_4-\frac{1}{2}x_8^2\mathbf{x}_3\right]\right.$$
$$\left.-\frac{3}{8\pi V\epsilon_l^2}\mathbf{x}_7x_6+\frac{1}{2V^2\epsilon_l^2}\mathbf{x}_3x_6^2\right] . \tag{20}$$

Owing to the fact that an ellipsoid possesses a considerable amount of symmetry several of the integrals appearing in Eq. (20) vanish and only those terms with integrands containing even powers of the variables n_x, n_y, and n_z persist. Consequently we may with impunity ignore all other terms.

It is possible to work with all the components of the torque vector $\mathbf{T}$ appearing in Eq. (20) but, for our purposes, it is sufficient to consider just the x component alone and the properties of the others will follow from symmetry. In evaluating Eq. (20) for the x component the following three integrals appear and may be expressed in terms of elementary functions:

$$\int da_s\frac{n_x^2}{h_\rho}=\int da_s\frac{n_y^2}{h_\rho}=\int da_s\frac{n_z^2}{h_\rho}=\frac{4\pi}{3}bc . \tag{21}$$

In addition to the above three relatively simple integrals we also obtain

$$J'\equiv\int da_s\frac{n_y^2n_z^2}{h_\rho^3}=\frac{bc}{a^2}\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\frac{\sin^5\theta\cos^2\phi\sin^2\phi}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta\cos^2\phi}{b^2}+\frac{\sin^2\theta\sin^2\phi}{c^2}}\equiv\frac{bc}{a^2}J . \tag{22}$$

The explicit details regarding the computation of this integral have been relegated to the appendix.

Considering only the nonvanishing integrals and the fact that all three components may be obtained by symmetry from the x component the torque vector can be expressed as follows:

$$\mathbf{T}=\left[\frac{\epsilon_l}{4\pi}\right]\int da_s\left[\mathbf{r}\times\mathbf{E}(\mathbf{E}\cdot\mathbf{n}_s)-\frac{1}{2}|\mathbf{E}|^2(\mathbf{r}\times\mathbf{n}_s)\right]$$
$$\equiv\mathbf{T}_1+\mathbf{T}_2+\mathbf{T}_3 , \tag{23}$$

$$\mathbf{T}_1=\frac{1}{V\epsilon_l}\int da_s(\mathbf{r}\times\mathbf{E}_0)(\mathbf{n}_s\cdot\mathbf{p}) , \tag{24}$$

$$\mathbf{T}_2 = -\frac{3}{8\pi V \epsilon_l^2}\int da_s(\mathbf{r}\times \mathbf{A}\cdot\mathbf{p})(\mathbf{n}_s\cdot\mathbf{p}) \ , \tag{25}$$

$$\mathbf{T}_3 = -\frac{1}{2V^2\epsilon_l^2}\int da_s(\mathbf{r}\times\mathbf{n}_s)(\mathbf{n}_s\cdot\mathbf{p})^2 \ . \tag{26}$$

In the case of a body that possess a shape such that the vectors $\mathbf{r}$ and $\mathbf{n}_s$ are parallel (for example, in the case of a sphere) $\mathbf{T}_2$ must vanish. For the case of a sphere where Eq. (16) is valid the vector $\mathbf{T}_3$ also vanishes leaving only $\mathbf{T}_1$ which is the source of the expression for the torque as calculated by the approximate effective dipole method.

Returning once again to the x component of the torque we obtain, by substituting Eqs. (21) and (22) into the x components of Eqs. (24), (25), and (26),

$$T_{1x} = \frac{abc}{3V}(E_{0z}p_y - E_0 y p_z) \ , \tag{27}$$

$$T_{2x} = \frac{abc}{8\pi\epsilon_l V}p_y p_z(A_2 - A_3) \ , \tag{28}$$

$$T_{3x} = \frac{a^3}{4\pi V^2}p_y p_z\left[\frac{1}{c^2} - \frac{1}{b^2}\right]J' \ . \tag{29}$$

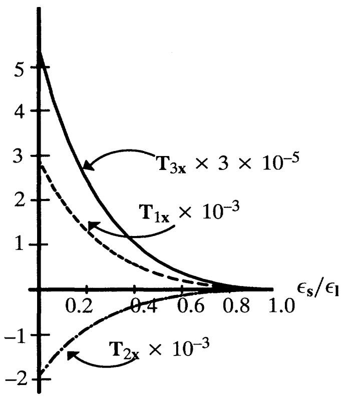

FIG. 2. Graph showing the relative magnitudes of the three terms T_{1x}, T_{2x}, and T_{3x} appearing in Eq. (30) as a function of the dielectric ratio. Ellipsoidal axes ratio used: $a:b:c = 8:4:2$

Introducing the results obtained above into Eq. (23) along with the expression for the integral J' computed in the Appendix the final equation for the x component of the torque is given by

$$T_x = \frac{1}{4\pi}(E_{0z}p_y - E_{0y}p_z) + \frac{3}{32\pi^2\epsilon_l} + (A_2 - A_3)p_y p_z - \frac{3bcP_3 - b^2c^2 - a^2c^2 - a^2b^2}{4\pi V a^2(b^2 - c^2)}p_y p_z$$
$$\equiv T_{1x} + T_{2x} + T_{3x} \ . \tag{30}$$

It is easy to see that the first term of this equation is indeed the result that is obtained from the effective-dipole-moment method (scaled by a factor of $1/4\pi$). There are, however, additional terms that appear, even in the nonlossy system, that we are considering in this paper. Before we comment on these terms we present a graphical comparison of the relative magnitudes to convince ourselves that these terms are indeed significant quantities.

III. GRAPHICAL COMPARISON

In Fig. 2 we plot the three terms appearing in Eq. (30) as a function of the ratio of the dielectric constants of the solid to that of the liquid (ϵ_s/ϵ_l). The dielectric constant of the liquid is taken to be 80. It is easy to see that, within the Maxwell-stress-tensor model, the contributions T_{1x} and T_{2x} dominate as $\epsilon_l \gg \epsilon_s$ and are much larger than the effective dipolar torque.

IV. CONCLUSIONS

In the classical theory of the torque exerted by a rotating electrical field (as discussed in Ref. [8] the energy of the particle is first computed in the form of an integral over its volume. The result of this calculation is an expression that involves the field within the particle and the angle it makes with the externally applied field. It is the differentiation of this energy expression with respect to this angle that produces the torque equation (1). In this expression the details of the surface play a very minor role since they, basically, only provide the limits of the volume integration. In the Maxwell-stress-tensor method it is the field outside the particle that plays the crucial role and the integration over the surface of the particle is sensitive to the actual geometry of the particle. The two terms $\mathbf{T}_2$ and $\mathbf{T}_3$ shown in Eqs. (25) and (26) result from the fact that on the surface of an ellipsoidal particle the vectors $\mathbf{n}_s$ and $A\cdot\mathbf{p}$ are not at all points parallel to $\mathbf{r}$.

Despite the fact that the result we have obtained from the Maxwell-stress-tensor method is different from that obtained from the traditional effective dipole method it is possible to recast our result into a similar form. This can be achieved if a different expression for the effective dipole moment is adopted. If we write the terms of T_{2x} and T_{3x} in Eqs. (28) and (29) explicitly and in the first term replace p_z by its definition in terms of E_{0z} obtained from Eq. (2) while in the second term replace p_y in an analogous manner by E_{0y}, we obtain

$$T_x = P_y E_{0z} - P_z E_{0y} \ , \tag{31}$$

$$P_y \equiv \frac{1}{4\pi}p_y + \frac{abc}{4\pi}\frac{A_2(\epsilon_s - \epsilon_l)p_y}{abc(\epsilon_s - \epsilon_l)A_3 + 2\epsilon_l} + \frac{2a^4b\epsilon_l(\epsilon_s - \epsilon_l)J'p_y}{3V^2abc(\epsilon_s - \epsilon_l)A_3 + 2\epsilon_l} \ , \tag{32}$$

$$P_z \equiv \frac{1}{4\pi}p_z + \frac{abc}{4\pi}\frac{A_3(\epsilon_s-\epsilon_l)p_z}{abc(\epsilon_s-\epsilon_l)A_2+2\epsilon_l} + \frac{2a^4c\epsilon_l(\epsilon_s-\epsilon_l)J'p_z}{3V^2abc(\epsilon_s-\epsilon_l)A_2+2\epsilon_l} . \tag{33}$$

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APPENDIX

The computation of the angular integral J is most conveniently carried out in the first octant:

$$J=8\int_0^{\pi/2}d\phi\int_0^{\pi/2}d\theta\frac{\sin^5\theta\cos^2\phi\sin^2\phi}{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta\cos^2\phi}{b^2}+\frac{\sin^2\theta\sin^2\phi}{c^2}} . \tag{A1}$$

The integration over the variable ϕ can be carried out exactly for the fundamental ellipsoid in which the three axes are unequal and the result is given by

$$I\equiv\frac{\pi[2k_2-p_1-2\sqrt{k_2(k_2-p_1)}}{4p_1^2} ,$$
$$p_1\equiv\left[\frac{1}{c^2}-\frac{1}{b_2}\right]\sin^2\theta , \tag{A2}$$
$$k_2\equiv\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{c^2} .$$

The final integration over the variable θ yields the quantity J and it can be written in a compact form as follows:

$$J=-\frac{4\pi b^2c^2(3bcP_3-b^2c^2-a^2c^2-a^2b^2)}{3a^2(b^2-c^2)^2} , \tag{A3}$$

where

$$P_3=\int_0^1 dx\sqrt{(b^2-a^2)x^2+a^2}\sqrt{(c^2-a^2)x^2+a^2} . \tag{A4}$$

The integral P_3 has to be calculated numerically.

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